TEST CODE **02234020**

FORM TP 2010230

MAY/JUNE 2010

CARIBBEAN EXAMINATIONS COUNCIL

ADVANCED PROFICIENCY EXAMINATION

PURE MATHEMATICS

UNIT 2 – PAPER 02

ANALYSIS, MATRICES AND COMPLEX NUMBERS

2 ½ hours

26 MAY 2010 (p.m.)

This examination paper consists of **THREE** sections: Module 1, Module 2 and Module 3.

Each section consists of 2 questions.
The maximum mark for each Module is 50.
The maximum mark for this examination is 150.
This examination consists of 6 printed pages.

INSTRUCTIONS TO CANDIDATES

1. **DO NOT** open this examination paper until instructed to do so.
2. Answer **ALL** questions from the **THREE** sections.
3. Write your solutions, with full working, in the answer booklet provided.
4. Unless otherwise stated in the question, any numerical answer that is not exact **MUST** be written correct to three significant figures.

Examination Materials Permitted

Graph paper (provided)
Mathematical formulae and tables (provided) – **Revised 2009**
Mathematical instruments
Silent, non-programmable, electronic calculator

SECTION A (Module 1)

Answer BOTH questions.

1. (a) The temperature of water, $x°$ C, in an insulated tank at time, t hours, may be modelled by the equation $x = 65 + 8e^{-0.02t}$. Determine the

(i) initial temperature of the water in the tank **[2 marks]**

(ii) temperature at which the water in the tank will eventually stabilize **[2 marks]**

(iii) time when the temperature of the water in the tank is 70° C. **[4 marks]**

(b) (i) Given that $y = e^{\tan^{-1}(2x)}$, where $-\frac{1}{2}\pi < \tan^{-1}(2x) < \frac{1}{2}\pi$, show that

$$(1 + 4x^2)\frac{dy}{dx} = 2y.$$ **[4 marks]**

(ii) Hence, show that $(1 + 4x^2)^2 \frac{d^2y}{dx^2} = 4y(1 - 4x)$. **[4 marks]**

(c) Determine $\int \frac{4}{e^x + 1}\,dx$

(i) by using the substitution $u = e^x$ **[6 marks]**

(ii) by first multiplying both the numerator and denominator of the integrand $\frac{4}{e^x + 1}$ by e^{-x} before integrating. **[3 marks]**

Total 25 marks

2. (a) (i) Given that n is a positive integer, find $\frac{d}{dx}[x(\ln x)^n]$. **[4 marks]**

(ii) Hence, or otherwise, derive the reduction formula $I_n = x(\ln x)^n - nI_{n-1}$, where $I_n = \int (\ln x)^n\,dx$. **[4 marks]**

(iii) Use the reduction formula in (a) (ii) above to determine $\int (\ln x)^3\,dx$. **[6 marks]**

(b) The amount of salt, y kg, that dissolves in a tank of water at time t minutes satisfies the differential equation $\frac{dy}{dt} + \frac{2y}{t+10} = 3$.

(i) Using a suitable integrating factor, show that the general solution of this differential equation is $y = t + 10 + \frac{c}{(t+10)^2}$, where c is an arbitrary constant. **[7 marks]**

(ii) Given that the tank initially contains 5 kg of salt in the liquid, calculate the amount of salt that dissolves in the tank of water at $t = 15$. **[4 marks]**

Total 25 marks

SECTION B (Module 2)

Answer BOTH questions.

3. (a) The first four terms of a sequence are

$2 \times 3, \quad 5 \times 5, \quad 8 \times 7, \quad 11 \times 9.$

(i) Express, in terms of r, the r th term of the sequence. **[2 marks]**

(ii) If S_n denotes the series formed by summing the first n terms of the sequence, find S_n in terms of n. **[5 marks]**

(b) The 9th term of an A.P. is three times the 3rd term and the sum of the first 10 terms is 110. Find the first term a **and** the common difference d. **[6 marks]**

(c) (i) Use the binominal theorem to expand $(1 + 2x)^{\frac{1}{2}}$ as far as the term in x^3, stating the values of x for which the expansion is valid. **[5 marks]**

(ii) Prove that $\frac{x}{1 + x + \sqrt{1+2x}} = \frac{1}{x}(1 + x - \sqrt{1+2x})$ for $x > 0$. **[4 marks]**

(iii) Hence, or otherwise, show that, if x is small so that the term in x^3 and higher powers of x can be neglected, the expansion in (c) (ii) above is approximately equal to

$\frac{1}{2}x(1 - x)$. **[3 marks]**

Total 25 marks

4. (a) (i) By expressing ${}^nC_r$ and ${}^nC_{r-1}$ in terms of factorials, prove that ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$.

[6 marks]

(ii) a) Given that r is a positive integer and $f(r) = \frac{1}{r!}$, show that

$$f(r) - f(r+1) = \frac{r}{(r+1)!}$$

[3 marks]

b) Hence, or otherwise, find the sum

$$S_n = \sum_{r=1}^{n} \frac{r}{(r+1)!}.$$

[5 marks]

c) Deduce the sum to infinity of S_n in (ii) b) above. **[2 marks]**

(b) (i) Show that the function $f(x) = x^3 - 6x + 4$ has a root x in the closed interval [0, 1].

[5 marks]

(ii) By taking 0.6 as a first approximation of x_1 in the interval [0, 1], use the Newton-Raphson method to obtain a second approximation x_2 in the interval [0, 1].

[4 marks]

Total 25 marks

SECTION C (Module 3)

Answer BOTH questions.

5. (a) Calculate

(i) the number of different permutations of the 8 letters of the word **SYLLABUS** **[3 marks]**

(ii) the number of different selections of 5 letters which can be made from the letters of the word **SYLLABUS**. **[5 marks]**

(b) The events A and B are such that $P(A) = 0.4$, $P(B) = 0.45$ and $P(A \cup B) = 0.68$.

(i) Find $P(A \cap B)$. **[3 marks]**

(ii) Stating a reason in each case, determine whether or not the events A and B are

a) mutually exclusive **[3 marks]**

b) independent. **[3 marks]**

(c) (i) Express the complex number $(2 + 3\mathrm{i}) + \dfrac{\mathrm{i} - 1}{\mathrm{i} + 1}$ in the form $a + \mathrm{i}b$, where a and b are both real numbers. **[4 marks]**

(ii) Given that $1 - \mathrm{i}$ is the root of the equation $z^3 + z^2 - 4z + 6 = 0$, find the remaining roots. **[4 marks]**

Total 25 marks

6. (a) A system of equations is given by

$$\begin{aligned} x + y + z &= 0 \\ 2x + y - z &= -1 \\ x + 2y + 4z &= k \end{aligned}$$

where k is a real number.

(i) Write the augmented matrix of the system. **[2 marks]**

(ii) Reduce the augmented matrix to echelon form. **[3 marks]**

(iii) Deduce the value of k for which the system is consistent. **[2 marks]**

(iv) Find ALL solutions corresponding to the value of k obtained in (iii) above. **[4 marks]**

(b) Given $A = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

(i) Find

a) A^2 **[4 marks]**

b) $B = 3I + A - A^2$ **[4 marks]**

(ii) Calculate AB. **[4 marks]**

(iii) Deduce the inverse, A^{-1}, of the matrix A. **[2 marks]**

Total 25 marks

END OF TEST