TEST CODE **22234020**

FORM TP 2008243

MAY/JUNE 2008

CARIBBEAN EXAMINATIONS COUNCIL

ADVANCED PROFICIENCY EXAMINATION

PURE MATHEMATICS

UNIT 2 – PAPER 02

ANALYSIS, MATRICES AND COMPLEX NUMBERS

2 ½ hours

15 JULY 2008 (p.m.)

This examination paper consists of **THREE** sections: Module 1, Module 2 and Module 3.

Each section consists of 2 questions.
The maximum mark for each Module is 50.
The maximum mark for this examination is 150.
This examination consists of 5 printed pages.

INSTRUCTIONS TO CANDIDATES

1. **DO NOT** open this examination paper until instructed to do so.
2. Answer **ALL** questions from the **THREE** sections.
3. Write your solutions, with full working, in the answer booklet provided.
4. Unless otherwise stated in the question, any numerical answer that is not exact **MUST** be written correct to three significant figures.

Examination Materials Permitted

Graph paper (provided)
Mathematical formulae and tables (provided) – **Revised 2008**
Mathematical instruments
Silent, non-programmable, electronic calculator

SECTION A (Module 1)

Answer BOTH questions.

1. (a) Find the **exact** values of x such that

$$e^x + 7e^{-x} = 8.$$ **[5 marks]**

(b) Given that $u = e^{2x} + e^{-2x}$, $v = e^{2x} - e^{-2x}$, show that

$$\frac{d^2u}{dx^2} + \frac{d^2v}{dx^2} = 2\left(\frac{du}{dx} + \frac{dv}{dx}\right).$$ **[7 marks]**

(c) (i) Differentiate with respect to x

$$(x \ln x) \sin^{-1} 2x.$$ **[4 marks]**

(ii) A curve **C** has parametric equations

$$x = 3t^2 + 5, \quad y = 2t^3 + 6t.$$

a) Show that $\frac{dy}{dx} = t + \frac{1}{t}$. **[3 marks]**

b) Show that **C** has points of inflexion at (8, 8) and (8, –8). **[6 marks]**

Total 25 marks

2. (a) (i) Find $\int \frac{1}{x} \ln x \, dx$. **[3 marks]**

(ii) Solve the differential equation

$$x^2 \frac{dy}{dx} + xy = \ln x.$$ **[5 marks]**

(b) (i) Find the values of the constants m and n, given that $y = m \cos x + n \sin x$ satisfies the differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 10 \sin x.$$ **[5 marks]**

(ii) Hence, find the general solution of the differential equation. **[3 marks]**

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(c) (i) Express $\dfrac{2 + 3x - x^2}{(x - 1)(x^2 + 1)}$ in partial fractions. **[6 marks]**

(ii) Hence, find $\displaystyle\int \frac{2 + 3x - x^2}{(x - 1)(x^2 + 1)}\,dx$. **[3 marks]**

Total 25 marks

SECTION B (Module 2)

Answer BOTH questions.

3\. (a) (i) Let $\mathbf{S} = \sum\limits_{r=1}^{n} r$ and $\mathbf{T} = \sum\limits_{r=1}^{n} (n + 1 - r)$.

a) Show that $\mathbf{T} = \mathbf{S}$. **[2 marks]**

b) Deduce that $\mathbf{S} = \dfrac{1}{2} n (n + 1)$. **[3 marks]**

(ii) Use the principle of mathematical induction to prove that

$$\sum_{r=1}^{n} r^2 = \frac{1}{6} n (n + 1)(2n + 1).$$ **[7 marks]**

(iii) Hence, prove that $\sum\limits_{r=1}^{n} 2r\,(3r + 1) = 2n (n + 1)^2$. **[4 marks]**

(b) (i) Show that the equation $x^3 + 3x^2 + 6x - 3 = 0$ has a root α between 0 and 1. **[2 marks]**

(ii) Prove that α is the only real root. **[3 marks]**

(iii) Using TWO iterations of the Newton-Raphson method, find α correct to 2 decimal places. **[4 marks]**

Total 25 marks

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4. (a) The sequence $\{a_n\}$ of **positive numbers** is defined by

$$a_{n+1} = \frac{4(1 + a_n)}{4 + a_n}, \; a_1 = \frac{3}{2}.$$

(i) Find a_2 and a_3. **[2 marks]**

(ii) Express $a_{n+1} - 2$ in terms of a_n. **[2 marks]**

(iii) Given that $a_n < 2$ for all n, show that

a) $a_{n+1} < 2$ **[3 marks]**

b) $a_n < a_{n+1}$. **[6 marks]**

(b) Find the term independent of x in the binomial expansion of $(x^2 - \frac{6}{x^3})^{15}$.

[You may leave your answer in the form of factorials and powers, for example, $\frac{15!}{2!}$ x 8^5.]

[6 marks]

(c) Use the binomial theorem to find the difference between 2^{10} and $(2.002)^{10}$ correct to 5 decimal places. **[6 marks]**

Total 25 marks

SECTION C (Module 3)

Answer BOTH questions.

5. (a) Four-digit numbers are formed from the digits 1, 2, 3, 4, 7, 9.

(i) How many 4-digit numbers can be formed if

a) the digits, 1, 2, 3, 4, 7, 9, can all be repeated? **[2 marks]**

b) none of the digits, 1, 2, 3, 4, 7, 9, can be repeated? **[2 marks]**

(ii) Calculate the probability that a 4-digit number in (a) (i) b) above is even. **[3 marks]**

(b) A father and son practise shooting at basketball, and score when the ball hits the basket. The son scores 75% of the time and the father scores 4 out of 7 tries. If EACH takes one shot at the basket, calculate the probability that only ONE of them scores. **[6 marks]**

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(c) (i) Find the values of $h, k \in \mathbf{R}$ such that $3 + 4i$ is a root of the quadratic equation

$$z^2 + hz + k = 0.$$ **[6 marks]**

(ii) Use De Moivre's theorem for $(\cos\theta + i\sin\theta)^3$ to show that

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta.$$ **[6 marks]**

Total 25 marks

6. (a) Solve for x the equation

$$\begin{vmatrix} 1 & 1 & 1 \\ x & 2 & 1 \\ x^3 & 8 & 1 \end{vmatrix} = 0.$$ **[12 marks]**

(b) The Popular Taxi Service in a certain city provides transportation for tours of the city using cars, coaches and buses. Selection of vehicles for tours of distances (in km) is as follows:

x cars, $2y$ coaches and $3z$ buses cover 34 km tours.
$2x$ cars, $3y$ coaches and $4z$ buses cover 49 km tours.
$3x$ cars, $4y$ coaches and $6z$ buses cover 71 km tours.

(i) Express the information above as a matrix equation

$$\mathbf{AX} = \mathbf{Y}$$

where **A** is 3 x 3 matrix, **X** and **Y** are 3 x 1 matrices with

$$\mathbf{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$ **[3 marks]**

(ii) Let $\mathbf{B} = \begin{pmatrix} -4 & 0 & 2 \\ 0 & 6 & -4 \\ 2 & -4 & 2 \end{pmatrix}$.

a) Calculate **AB**. **[3 marks]**

b) Deduce the inverse $\mathbf{A}^{-1}$ of **A**. **[3 marks]**

(iii) Hence, or otherwise, determine the number of cars and buses used in the 34 km tours. **[4 marks]**

Total 25 marks

END OF TEST