2013

SECTION A (Module 1)

Answer BOTH questions.

1. (a) Calculate the gradient of the curve $\ln(x^2y) - \sin y = 3x - 2y$ at the point (1, 0). **[5 marks]**

   (b) Let $f(x, y, z) = 3yz^2 - e^{4x}\cos 4z - 3y^2 - 4 = 0$.

   Given that $\dfrac{\partial z}{\partial y} = -\dfrac{\partial f / \partial y}{\partial f / \partial z}$, determine $\dfrac{\partial z}{\partial y}$ in terms of x, y and z. **[5 marks]**

   (c) Use de Moivre's theorem to prove that

   $$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta.$$ **[6 marks]**

   (d) (i) Write the complex number $z = (-1 + i)^7$ in the form $re^{i\theta}$, where $r = |z|$ and $\theta = \arg z$. **[3 marks]**

   (ii) Hence, prove that $(-1 + i)^7 = -8(1 + i)$. **[6 marks]**

Total 25 marks

2. (a) (i) Determine $\int \sin x \cos 2x \, dx$. **[5 marks]**

   (ii) Hence, calculate $\int_0^{\frac{\pi}{2}} \sin x \cos 2x \, dx$. **[2 marks]**

   (b) Let $f(x) = x\,|x| = \begin{cases} x^2 & ; x \geq 0 \\ -x^2 & ; x < 0 \end{cases}$.

   Use the trapezium rule with four intervals to calculate the area between $f(x)$ and the x-axis for the domain $-0.75 \leq x \leq 2.25$. **[5 marks]**

   (c) (i) Show that $\dfrac{2x^2 + 4}{(x^2 + 4)^2} = \dfrac{2}{x^2 + 4} - \dfrac{4}{(x^2 + 4)^2}$. **[6 marks]**

   (ii) Hence, find $\int \dfrac{2x^2 + 4}{(x^2 + 4)^2} \, dx$. Use the substitution $x = 2\tan\theta$. **[7 marks]**

Total 25 marks

SECTION B (Module 2)

Answer BOTH questions.

3. (a) The sequence $\{a_n\}$ is defined by $a_1 = 1$, $a_{n+1} = 4 + 2\sqrt[3]{a_n}$.

Use mathematical induction to prove that $1 \leq a_n \leq 8$ for all n in the set of positive integers. **[6 marks]**

(b) Let $k > 0$ and let $f(k) = \frac{1}{k^2}$.

(i) Show that

a) $f(k) - f(k+1) = \dfrac{2k+1}{k^2 (k+1)^2}$. **[3 marks]**

b) $\sum\limits_{k=1}^{n} \left(\dfrac{1}{k^2} - \dfrac{1}{(k+1)^2} \right) = 1 - \dfrac{1}{(n+1)^2}$. **[5 marks]**

(iii) Hence, or otherwise, prove that

$$\sum_{k=1}^{\infty} \frac{2k+1}{k^2 (k+1)^2} = 1.$$ **[3 marks]**

(c) (i) Obtain the first four non-zero terms of the Taylor Series expansion of $\cos x$ in ascending powers of $(x - \frac{\pi}{4})$. **[5 marks]**

(ii) Hence, calculate an approximation to $\cos(\frac{\pi}{16})$. **[3 marks]**

Total 25 marks

4. (a) (i) Obtain the binomial expansion of

$$\sqrt[4]{(1+x)} + \sqrt[4]{(1-x)}$$

up to the term containing x^2. **[4 marks]**

(ii) Hence, by letting $x = \frac{1}{16}$, compute an approximation of $\sqrt[4]{17} + \sqrt[4]{15}$ to four decimal places. **[4 marks]**

(b) Show that the coefficient of the x^5 term of the product $(x+2)^5(x-2)^4$ is 96. **[7 marks]**

(c) (i) Use the Intermediate Value Theorem to prove that $x^3 = 25$ has at least one root in the interval [2, 3]. **[3 marks]**

(ii) The table below shows the results of the first four iterations in the estimation of the root of $f(x) = x^3 - 25 = 0$ using interval bisection.

The procedure used $a = 2$ and $b = 3$ as the starting points and p_n is the estimate of the root for the n^{th} iteration.

n	a_n	b_n	p_n	$f(p_n)$
1	2	3	2.5	–9.375
2	2.5	3	2.75	–4.2031
3	2.75	3	2.875	–1.2363
4	2.875	3	2.9375	0.3474
5				
6				
......				
......				

Complete the table to obtain an approximation of the root of the equation $x^3 = 25$ correct to 2 decimal places. **[7 marks]**

Total 25 marks

SECTION C (Module 3)

Answer BOTH questions.

5. (a) Three letters from the word BRIDGE are selected one after the other without replacement. When a letter is selected, it is classified as either a vowel (V) or a consonant (C).

Use a tree diagram to show the possible outcomes (vowel or consonant) of the THREE selections. Show all probabilities on the diagram. **[7 marks]**

(b) (i) The augmented matrix for a system of three linear equations with variables x, y and z respectively is

$$A = \left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ -5 & 1 & 1 & 2 \\ 1 & -5 & 3 & 3 \end{array}\right)$$

By reducing the augmented matrix to echelon form, determine whether or not the system of linear equations is consistent. **[5 marks]**

(ii) The augmented matrix for another system is formed by replacing the THIRD row of A in (i) above with $(1 \ -5 \quad 5 \mid 3)$.

Determine whether the solution of the new system is unique. **Give a reason for your answer.** **[5 marks]**

(c) A country, X, has three airports (A, B, C). The percentage of travellers that use each of the airports is 45%, 30% and 25% respectively. Given that a traveller has a weapon in his/her possession, the probability of being caught is, 0.7, 0.9 and 0.85 for airports A, B, and C respectively.

Let the event that:

- the traveller is caught be denoted by D, and
- the event that airport A, B, or C is used be denoted by A, B, and C respectively.

(i) What is the probability that a traveller using an airport in Country X is caught with a weapon? **[5 marks]**

(ii) On a particular day, a traveller was caught carrying a weapon at an airport in Country X. What is the probability that the traveller used airport C? **[3 marks]**

Total 25 marks

6. (a) (i) Obtain the general solution of the differential equation

$$\cos x \frac{dy}{dx} + y \sin x = 2x \cos^2 x.$$ **[7 marks]**

(ii) Hence, given that $y = \frac{15\sqrt{2}\pi^2}{32}$, when $x = \frac{\pi}{4}$, determine the constant of the integration. **[5 marks]**

(b) The general solution of the differential equation

$$y'' + 2y' + 5y = 4 \sin 2t$$

is $y = CF + PI$, where CF is the complementary function and PI is a particular integral.

(i) a) Calculate the roots of

$\lambda^2 + 2\lambda + 5 = 0$, the auxiliary equation. **[2 marks]**

b) Hence, obtain the complementary function (CF), the general solution of

$y'' + 2y' + 5y = 0.$ **[3 marks]**

(ii) Given that the form of the particular integral (PI) is

$$u_p(t) = A \cos 2t + B \sin 2t,$$

Show that $A = -\frac{16}{17}$ and $B = \frac{4}{17}$. **[3 marks]**

(iii) Given that $y(0) = 0.04$ and $y'(0) = 0$, obtain the general solution of the differential equation. **[5 marks]**

Total 25 marks

END OF TEST

IF YOU FINISH BEFORE TIME IS CALLED, CHECK YOUR WORK ON THIS TEST.