TEST CODE **02234020**

FORM TP 2011234

MAY/JUNE 2011

CARIBBEAN EXAMINATIONS COUNCIL

ADVANCED PROFICIENCY EXAMINATION

PURE MATHEMATICS

UNIT 2 – PAPER 02

ANALYSIS, MATRICES AND COMPLEX NUMBERS

2 ½ hours

25 MAY 2011 (p.m.)

This examination paper consists of **THREE** sections: Module 1, Module 2 and Module 3.

Each section consists of 2 questions.
The maximum mark for each Module is 50.
The maximum mark for this examination is 150.
This examination consists of 7 printed pages.

INSTRUCTIONS TO CANDIDATES

1. **DO NOT** open this examination paper until instructed to do so.
2. Answer **ALL** questions from the **THREE** sections.
3. Write your solutions, with full working, in the answer booklet provided.
4. Unless otherwise stated in the question, any numerical answer that is not exact **MUST** be written correct to three significant figures.

Examination Materials Permitted

Graph paper (provided)
Mathematical formulae and tables (provided) – **Revised 2010**
Mathematical instruments
Silent, non-programmable, electronic calculator

SECTION A (Module 1)

Answer BOTH questions.

1. (a) Find $\frac{dy}{dx}$ if

(i) $x^2 + y^2 - 2x + 2y - 14 = 0$ **[3 marks]**

(ii) $y = e^{\cos x}$ **[3 marks]**

(iii) $y = \cos^2 6x + \sin^2 8x.$ **[3 marks]**

(b) Let $y = x \sin \frac{1}{x}, x \neq 0.$

Show that

(i) $x \frac{dy}{dx} = y - \cos\left(\frac{1}{x}\right)$ **[3 marks]**

(ii) $x^4 \frac{d^2y}{dx^2} + y = 0.$ **[3 marks]**

(c) A curve is given by the parametric equations $x = \sqrt{t},\ y - t = \frac{1}{\sqrt{t}}$.

(i) Find the gradient of the tangent to the curve at the point where $t = 4$. **[7 marks]**

(ii) Find the equation of the tangent to the curve at the point where $t = 4$. **[3 marks]**

Total 25 marks

2. (a) Let $F_n(x) = \frac{1}{n!} \int_0^x t^n e^{-t}\, dt$.

(i) Find $F_0(x)$ and $F_n(0)$, given that $0! = 1$. **[3 marks]**

(ii) Show that $F_n(x) = F_{n-1}(x) - \frac{1}{n!} x^n e^{-x}$. **[7 marks]**

(iii) Hence, show that if M is an integer greater than 1, then

$$e^x F_M(x) = -\left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^M}{M!} \right) + (e^x - 1).$$ **[4 marks]**

(b) (i) Express $\frac{2x^2 + 3}{(x^2 + 1)^2}$ in partial fractions. **[5 marks]**

(ii) Hence, find $\int \frac{2x^2 + 3}{(x^2 + 1)^2}\, dx$. **[6 marks]**

Total 25 marks

SECTION B (Module 2)

Answer BOTH questions.

3. (a) The sequence of positive terms, $\{x_n\}$, is defined by $x_{n+1} = x_n^2 + \frac{1}{4}$, $x_1 < \frac{1}{2}$, $n \geq 1$.

(i) Show, by mathematical induction, that $x_n < \frac{1}{2}$ for all positive integers n. **[5 marks]**

(ii) By considering $x_{n+1} - x_n$, show that $x_n < x_{n+1}$. **[3 marks]**

(b) (i) Find the constants A and B such that

$$\frac{2-3x}{(1-x)(1-2x)} \equiv \frac{A}{1-x} + \frac{B}{1-2x}.$$ **[3 marks]**

(ii) Obtain the **first** FOUR non-zero terms of the expansion of each of $(1-x)^{-1}$ and $(1-2x)^{-1}$ as power series of x in ascending order. **[4 marks]**

(iii) Find

a) the range of values of x for which the series expansion of

$$\frac{2-3x}{(1-x)(1-2x)}$$

is valid **[2 marks]**

b) the coefficient of x^n in (iii) a) above. **[2 marks]**

(iv) The sum, S_n, of the first n terms of a series is given by

$$S_n = n(3n-4).$$

Show that the series is an Arithmetic Progression (A.P.) with common difference 6. **[6 marks]**

Total 25 marks

4. (a) A Geometric Progression (G.P.) with first term a and common ratio r, $0 < r < 1$, is such that the sum of the first three terms is $\frac{26}{3}$ and their product is 8.

(i) Show that $r + 1 + \frac{1}{r} = \frac{13}{3}$. **[4 marks]**

(ii) Hence, find

a) the value of r **[4 marks]**

b) the value of a **[1 mark]**

c) the sum to infinity of the G.P. **[2 marks]**

(b) Expand

$$\frac{2}{e^x + e^{-x}}, \quad |x| < 1$$

in ascending powers of x as far as the term in x^4. **[5 marks]**

(c) Let $f(r) = \frac{1}{r(r+1)}$, $r \in \mathbf{N}$.

(i) Express $f(r) - f(r + 1)$ in terms of r. **[3 marks]**

(ii) Hence, or otherwise, find

$$S_n = \sum_{r=1}^{n} \frac{3}{r(r+1)(r+2)}.$$ **[4 marks]**

(iii) Deduce the sum to infinity of the series in (c) (ii) above. **[2 marks]**

Total 25 marks

SECTION C (Module 3)

Answer BOTH questions.

5. (a) $\binom{n}{r}$ is defined as the number of ways of selecting r distinct objects from a given set of n distinct objects. From the definition, show that

(i) $\binom{n}{r} = \binom{n}{n-r}$ **[2 marks]**

(ii) $\binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1}$ **[4 marks]**

(iii) Hence, prove that

$$\left(\binom{8}{6} + \binom{8}{5}\right) \times \left(\binom{8}{3} + \binom{8}{2}\right)$$

is a perfect square. **[3 marks]**

(b) (i) Find the number of 5-digit numbers greater than 30 000 which can be formed with the digits, 1, 3, 5, 6 and 8, if no digit is repeated. **[3 marks]**

(ii) What is the probability of one of the numbers chosen in (b) (i) being even? **[5 marks]**

(c) (i) a) Show that $(1 - i)$ is one of the square roots of $-2i$. **[2 marks]**

b) Find the other square root. **[1 mark]**

(ii) Hence, find the roots of the quadratic equation

$$z^2 - (3 + 5i)\, z + (8i - 4) = 0.$$ **[5 marks]**

Total 25 marks

6. (a) The matrix $\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -3 & 2 \\ -1 & -3 & -2 \end{pmatrix}$.

(i) Show that $|\mathbf{A}| = 5$. **[3 marks]**

(ii) Matrix **A** is changed to form new matrices **B**, **C** and **D**. Write down the determinant of EACH of the new matrices, **giving a reason for your answer in EACH case**.

a) Matrix **B** is formed by interchanging row 1 and row 2 of matrix **A** and then interchanging column 1 and column 2 of the resulting matrix. **[2 marks]**

b) Row 1 of matrix **C** is formed by adding row 2 to row 1 of matrix **A**. The other rows remain unchanged. **[2 marks]**

c) Matrix **D** is formed by multiplying each element of matrix **A** by 5. **[2 marks]**

(b) Given the matrix $\mathbf{M} = \begin{pmatrix} 12 & -1 & 5 \\ 2 & -1 & 0 \\ -9 & 2 & -5 \end{pmatrix}$,

Find

(i) **AM** **[3 marks]**

(ii) the inverse, $\mathbf{A}^{-1}$, of **A**. **[2 marks]**

(c) (i) Write the system of equations

$$\begin{aligned} x + y + z &= 5 \\ 2x - 3y + 2z &= -10 \\ -x - 3y - 2z &= -11 \end{aligned}$$

in the form $\mathbf{A}x = \mathbf{b}$. **[1 mark]**

(ii) Show that $x = \mathbf{A}^{-1}\mathbf{b}$. **[2 marks]**

(iii) Hence, solve the system of equations. **[2 marks]**

(iv) a) Show that $(x, y, z) = (1, 1, 1)$ is a solution of the following system of equations:

$$\begin{aligned} x + y + z &= 3 \\ 2x + 2y + 2z &= 6 \\ 3x + 3y + 3z &= 9 \end{aligned}$$

[1 mark]

b) Hence, find the general solution of the system. **[5 marks]**

Total 25 marks

END OF TEST