FORM TP 2009237

TEST CODE **02234020**

MAY/JUNE 2009

CARIBBEAN EXAMINATIONS COUNCIL

ADVANCED PROFICIENCY EXAMINATION

PURE MATHEMATICS

UNIT 2 – PAPER 02

ANALYSIS, MATRICES AND COMPLEX NUMBERS

2 ½ hours

27 MAY 2009 (p.m.)

This examination paper consists of **THREE** sections: Module 1, Module 2 and Module 3.

Each section consists of 2 questions.
The maximum mark for each Module is 50.
The maximum mark for this examination is 150.
This examination consists of 5 printed pages.

INSTRUCTIONS TO CANDIDATES

1. **DO NOT** open this examination paper until instructed to do so.
2. Answer **ALL** questions from the **THREE** sections.
3. Write your solutions, with full working, in the answer booklet provided.
4. Unless otherwise stated in the question, any numerical answer that is not exact **MUST** be written correct to three significant figures.

Examination Materials Permitted

Graph paper (provided)
Mathematical formulae and tables (provided) – **Revised 2009**
Mathematical instruments
Silent, non-programmable, electronic calculator

SECTION A (Module 1)

Answer BOTH questions.

1. (a) Find $\frac{dy}{dx}$ if

   (i) $y = \sin^2 5x + \sin^2 3x + \cos^2 3x$ **[3 marks]**

   (ii) $y = \sqrt{\cos x^2}$ **[4 marks]**

   (iii) $y = x^x$. **[4 marks]**

   (b) (i) Given that $y = \cos^{-1} x$, where $0 \leq \cos^{-1} x \leq \pi$, prove that $\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$.

   [Note: $\cos^{-1}x \equiv \text{arc}\cos x$] **[7 marks]**

   (ii) The parametric equations of a curve are defined in terms of a parameter t by $y = \sqrt{1-t}$ and $x = \cos^{-1} t$, where $0 \leq t < 1$.

   a) Show that $\frac{dy}{dx} = \frac{\sqrt{1+t}}{2}$. **[4 marks]**

   b) Hence, find $\frac{d^2y}{dx^2}$ in terms of t, giving your answer in simplified form. **[3 marks]**

Total 25 marks

2. (a) Sketch the region whose area is defined by the integral $\int_0^1 \sqrt{1-x^2}\, dx$. **[3 marks]**

   (b) Using FIVE vertical strips, apply the trapezium rule to show that $\int_0^1 \sqrt{1-x^2}\, dx \approx 0.759$. **[6 marks]**

   (c) (i) Use integration by parts to show that, if $I = \int \sqrt{1-x^2}\, dx$, then

$$I = x\sqrt{1-x^2} - I + \int \frac{1}{\sqrt{1-x^2}}\, dx.$$ **[9 marks]**

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(ii) Deduce that $I = \dfrac{x\sqrt{1-x^2} + \sin^{-1}x}{2} + c$, where c is an arbitrary constant of integration.

[Note: $\cos^{-1}x \equiv \arccos x$] **[2 marks]**

(iii) Hence, find $\int_0^1 \sqrt{1-x^2}\,dx$. **[3 marks]**

(iv) Use the results in Parts (b) and (c) (iii) above to find an approximation to π. **[2 marks]**

Total 25 marks

SECTION B (Module 2)

Answer BOTH questions.

3. (a) A sequence $\{t_n\}$ is defined by the recurrence relation

$$t_{n+1} = t_n + 5,\ t_1 = 11 \text{ for all } n \in \mathbf{N}.$$

(i) Determine t_2, t_3 and t_4. **[3 marks]**

(ii) Express t_n in terms of n. **[5 marks]**

(b) Find the range of values of x for which the common ratio r of a convergent geometric series is $\dfrac{2x-3}{x+4}$. **[8 marks]**

(c) Let $f(r) = \dfrac{1}{r+1}$, $r \in \mathbf{N}$.

(i) Express $f(r) - f(r+1)$ in terms of r. **[3 marks]**

(ii) Hence, or otherwise, find

$$S_n = \sum_{r=1}^{n} \frac{4}{(r+1)(r+2)}.$$ **[4 marks]**

(iii) Deduce the sum to infinity of the series in (c) (ii) above. **[2 marks]**

Total 25 marks

4. (a) (i) Find $n \in \mathbb{N}$ such that $5({}^{n}C_2) = 2({}^{n+2}C_2)$. **[5 marks]**

(ii) The coefficient of x^2 in the expansion of

$$(1 + 2x)^5 \; (1 + px)^4$$

is -26. Find the possible values of the real number p. **[7 marks]**

(b) (i) Write down the first FOUR non-zero terms of the power series expansion of $\ln(1 + 2x)$, stating the range of values of x for which the series is valid. **[2 marks]**

(ii) Use Maclaurin's theorem to obtain the first THREE non-zero terms in the power series expansion in x of $\sin 2x$. **[7 marks]**

(iii) Hence, or otherwise, obtain the first THREE non-zero terms in the power series expansion in x of

$$\ln(1 + \sin 2x).$$ **[4 marks]**

Total 25 marks

SECTION C (Module 3)

Answer BOTH questions.

5. (a) A committee of 4 persons is to be chosen from 8 persons, including Mr Smith and his wife. Mr Smith will not join the committee without his wife, but his wife will join the committee without him.

Calculate the number of ways in which the committee of 4 persons can be formed. **[5 marks]**

(b) Two balls are drawn without replacement from a bag containing 12 balls numbered 1 to 12.

Find the probability that

(i) the numbers on BOTH balls are even **[4 marks]**

(ii) the number on one ball is odd and the number on the other ball is even. **[4 marks]**

(c) (i) Find complex numbers u = x + iy such that x and y are real numbers and

$$u^2 = -15 + 8i.$$ **[7 marks]**

(ii) Hence, or otherwise, solve for z the equation

$$z^2 - (3 + 2i)\, z + (5 + i) = 0.$$ **[5 marks]**

Total 25 marks

6. (a) Solve for x the equation

$$\begin{vmatrix} x-3 & 1 & -1 \\ 1 & x-5 & 1 \\ -1 & 1 & x-3 \end{vmatrix} = 0.$$ **[10 marks]**

(b) (i) Given the matrices

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & -2 & 4 \\ 1 & 3 & 9 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 30 & -12 & 2 \\ 5 & -8 & 3 \\ -5 & 4 & 1 \end{pmatrix},$$

a) find **AB** **[4 marks]**

b) hence deduce the inverse $\mathbf{A}^{-1}$ of the matrix **A**. **[3 marks]**

(ii) A system of equations is given by

$$x - y + z = 1$$

$$x - 2y + 4z = 5$$

$$x + 3y + 9z = 25.$$

a) Express the system in the form

$\mathbf{A}x = \mathbf{b}$, where **A** is a matrix and x and **b** are column vectors.

b) Hence, or otherwise, solve the system of equations. **[5 marks]**

Total 25 marks

END OF TEST