02234020/CAPE/MS/SPEC

CARIBBEAN EXAMINATIONS COUNCIL

ADVANCED PROFICIENCY EXAMINATION

PURE MATHEMATICS

SPECIMEN PAPER

UNIT 2

COMPLEX NUMBERS, ANALYSIS AND MATRICES

PAPER 02

SOLUTIONS AND MARK SCHEMES

SECTION A

(MODULE 1)

Question 1

(a) (i) $\frac{4-2\mathrm{i}}{1-3\mathrm{i}} = \frac{(4-2\mathrm{i})(1+3\mathrm{i})}{1-3\mathrm{i})(1+3\mathrm{i})}$

$= \frac{4+12\mathrm{i}-2\mathrm{i}-6\mathrm{i}^2}{1+9}$ (1 mark)

$= \frac{4+10\mathrm{i}+6}{10}$ (1 mark)

$= \frac{10+10\mathrm{i}}{10}$ (1 mark)

$= 1+\mathrm{i}$ (1 mark)

[4 marks]

(ii) arg is $\tan^{-1}(1)\,) = \frac{\pi}{4}$ (1 mark)

[1 mark]

(b) (i) Let $u = x + \mathrm{i}\,y$, where x, y are real nos.

$u^2 = -5 + 12\mathrm{i}$ □ $(x + \mathrm{i}y)^2 = -5 + 12\mathrm{i}$

□ $x^2 - y^2 + 2\mathrm{i}xy = -5 + 12\mathrm{i}$ (1 mark)

□ $x^2 - y^2 = -5, \quad 2xy = 12$ (1 mark)

□ $x^2 - \left(\frac{6}{x}\right)^2 = -5, \quad y = \frac{6}{x}$

□ $x^2 - \frac{36}{x^2} = -5$ (1 mark)

□ $(x^2)^2 + 5x^2 - 36 = 0$ (1 mark)

□ $(x^2 + 9)(x^2 - 4) = 0$ (1 mark)

□ $x^2 = -9(\text{inadmissable}), x^2 = 4$ (1 mark)

□ $x = \pm 2, y =$ □3 (1 mark)

□$= 2 - 3\mathrm{i}$ or $-2 + 3\mathrm{i}$ (1 mark)

[8 marks]

(b) (ii) $z^2 + \mathrm{i}z + (1 - 3\mathrm{i}) = 0$ □ $z = \frac{-\mathrm{i} \pm \sqrt{\mathrm{i}^2 - 4(1-3\mathrm{i})}}{2}$ (1 mark)

□ $z \frac{-\mathrm{i} \pm \sqrt{-1-4+12\mathrm{i}}}{2}$ (1 mark)

□ $z \frac{-\mathrm{i} \pm \sqrt{-5+12\mathrm{i}}}{2}$ (1 mark)

□ $z \frac{-\mathrm{i} \pm 2-3\mathrm{i}}{2}$ (1 mark)

□ $z \frac{2-4\mathrm{i}}{2}$ or $\frac{2+2\mathrm{i}}{2}$ (1 mark)

□ $z = 1 - 2\mathrm{i}$ or $-1 + \mathrm{i}$ (1 mark)

[6 marks]

(c) $(1 + 3\mathrm{i})z + (4 - 2\mathrm{i})z = 10 + 4\mathrm{i}$, and $z = a + \mathrm{i}b$

□ $(1 + 3\mathrm{i})(a + \mathrm{i}b) + (4 - 2\mathrm{i})(a - \mathrm{i}b) = 10 + 4\mathrm{i}$ (1 mark)

□ $(a - 3b) + \mathrm{i}(3a + b) + (4a - 2b) + \mathrm{i}(-4b - 2a) = 10 + 4\mathrm{i}$ (1 mark)

□ $a - 3b + 4a - 2b = 10$ and $3a + b - 4b - 2a = 4$ (1 mark)

□ $5a - 5b = 10$ and $a - 3b = 4$ (1 mark)

□ $a = 1, b = -1$ (1 mark)

□ $z = 1 - \mathrm{i}$ (1 mark)

[6 marks]

Total 25 marks

Specific Objectives (A) 1, 4, 5, 6, 7, 8.

Question 2

(a) Let I = $\int e^{3x} \sin 2x \, dx$

$= \frac{1}{3} e^{3x} \sin 2x - \int \frac{e^{3x}}{3} (2\cos 2x) \, dx$ (2 marks)

$= \frac{1}{3} e^{3x} \sin 2x - \frac{2}{3} \int e^{3x} (2\cos 2x) \, dx$

$= \frac{1}{3} e^{3x} \sin 2x - \frac{2}{3} \left| \frac{1}{3} e^{3x} \cos 2x + \int \frac{e^{3x}}{3} (2\sin 2x) \, dx \right|$ (2 marks)

$= \frac{1}{3} e^{3x} \sin 2x - \frac{2}{9} e^{3x} \cos 2x - \frac{4}{9} \int e^{3x} \sin 2x \, dx$

$= \frac{1}{3} e^{3x} \sin 2x - \frac{2}{9} e^{3x} \cos 2x - \frac{4}{9} I$ (1 mark)

□ $I + \frac{4}{9} I = \frac{1}{9} (3 \sin 2x - 2\cos 2x)$ (1 mark)

□ $I = \frac{1}{13} (3 \sin 2x - 2\cos 2x)$ + constant (1 mark)

[7 marks]

Alternatively

$\int e^{3x} e^{2ix} dx = \int e^{(3+2i)x} \, dx$ (2 marks)

□ Im $\left| \frac{e^{(3+2i)x}}{3+2i} \right|$ + constant (2 marks)

□ $\int e^{3x} \sin 2x \, dx = \mathrm{Im} \, \frac{(3-2i)}{13} e^{3x} (\cos 2x + i \sin 2x)$ (2 marks)

$\frac{e^{3x}}{13} (3 \sin 2x - 2\cos 2x)$ + const. (1 mark)

[7 marks]

(b) (i) a) $y = \tan^{-1}(3x)$ □ $\tan y = 3x$ (1 mark)

□ $\sec^2 y \, \frac{dy}{dx} = 3$

□ $\frac{dy}{dx} = \frac{3}{\sec^2 y}$ (1 mark)

□ $\frac{dy}{dx} = \frac{3}{1 + \tan^2 y}$ (1 mark)

□ $\frac{dy}{dx} = \frac{3}{1 + 9x^2}$ (1 mark)

[4 marks]

b) $\int \frac{x+2}{1 + 9x^2} dx = \int \frac{x}{1 + 9x^2} \, dx + 2 \int \frac{1}{1 + 9x^2} \, dx$ (1 mark)

$= \frac{1}{18} \ln (1 + 9x^2) + \frac{2}{3} \tan^{-1}(3x)$ + constant (3 marks)

[4 marks]

(b) (ii) $y = \frac{\ln(5x)}{x^2}$, Using the product rule: $y = \frac{1}{x^2}\ln(5x)$ (1 mark)

$\square \frac{dy}{dx} = -\frac{2}{x^3}\ln(5x) + \frac{1}{x^2} \times \frac{1}{x}$ (2 marks)

$= \frac{1-2\ln(5x)}{x^3}$ (1 mark)

$= \frac{1-\ln(25x^2)}{x^3}$ (1 mark)

[5 marks]

c) $f(x,y) = x^2 + y^2 - 2xy$

(i) $\frac{\partial f}{\partial x} = 2x - 2y$ $\frac{\partial f}{\partial y} = 2y - 2x$ [2 marks]

(ii) $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = x(2x - 2y) + y(2y - 2x)$

$= 2x^2 + 2y^2 - 4xy$ (1 mark)

$= 2(x^2 + y^2 - 2xy)$ (1 mark)

$= 2(f(x,y)$ (1 mark)

[3 marks]

Total 25 marks

Specific Objectives: (A) 13, (B) 1, 2, 5, 6, 8, (C) 4, 5, 6, 8

SECTION B

(MODULE 2)

Question 3

(a) (i) $$\frac{1}{(2r-1)(2r+1)} \equiv \frac{A}{2r-1} + \frac{B}{2r+1}$$

$\Rightarrow$ $1 = A(2r+1) + B(2r-1)$ (1 mark)

$\Rightarrow$ $0 = 2A + 2B$ and $A - B = 1$ (2 marks)

$\Rightarrow$ $A = \frac{1}{2}$ and $B = -\frac{1}{2}$ (2 marks)

[5 marks]

(ii) $$\mathrm{S} = \sum_{r=1}^{n} \frac{1}{(2r-1)(2r+1)} = \sum_{r=1}^{n} \frac{1}{2}\left(\frac{1}{2r-1} - \frac{1}{2r+1}\right)$$ (1 mark)

$$= \frac{1}{2}\left(\frac{1}{1} - \frac{1}{3}\right) + \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5}\right) + \frac{1}{2}\left(\frac{1}{5} - \frac{1}{7}\right) + \ldots + \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)$$

(3 marks)

$$= \frac{1}{2}\left(1 - \frac{1}{2n+1}\right)$$ (1 mark)

[5 marks]

(iii) As $n \to \infty$, $\frac{1}{2n+1} \to 0$ (2 marks)

Hence $S_\infty = \frac{1}{2}$ (1 mark)

[3 marks]

(b) (i) $\mathrm{S} = 1(2) + 2(5) + 3(8) + \ldots$

In each term, 1st factor is in the natural sequence and the second factor differs by 3 (1 mark)

$\Rightarrow$ the r^{th} term is $r(3r - 1)$ (1 mark)

[2 marks]

(ii) $$S_n = \sum_{r=1}^{n} r(3r-1)$$

for $n = 1$ $\quad S_1 = \sum_{r=1}^{1} r(3r-1) = 1 \times 2 = 2$

and $\quad 1^2\,(1 + 1) = 1 \times 2 = 2$ (1 mark)

hence, $\quad S_n = n^2\,(n+1)$ is true for $n = 1$ (1 mark)

Assume $\quad S_n = n^2\,(n+1)$ for $n = k \in \mathbb{N}$ (1 mark)

that is, $\quad S_k = k^2\,(k+1)$ (1 mark)

Then, $S_{k+1} = \sum_{r=1}^{k+1} r\,(3r-1) = S_k + (k+1)(3k+2)$ (1 mark)

$= k^2(k+1) + (k+1)(3k+2)$ (1 mark)

$= (k+1)\left[k^2 + 3k + 2\right]$ (1 mark)

$\Rightarrow \quad S_{k+1} = (k+1)\left[(k+1)(k+2)\right]$

$= (k+1)^2\left[(k+1)+1\right]$ (1 mark)

$\Rightarrow$ true for $n = k + 1$ whenever it is assumed true for $n = k$, (1 mark)

$\Rightarrow$ true for all $n \in \mathrm{N}$

$\Rightarrow \quad S_n = n^2(n+1)\ n$ □ ℕ. (1 mark)

[10 marks]

Total 25 marks

Specific Objectives: (B) 1, 3, 5, 6, 10

Question 4

(a) (i) Let $S \equiv \frac{1}{2} + \frac{1}{2^4} + \frac{1}{2^7} + \frac{1}{2^{10}} + \ldots$

$$\frac{\frac{1}{2^4}}{\frac{1}{2}} = \frac{\frac{1}{2^7}}{\frac{1}{2^4}}$$ (1 mark)

$$= \frac{1}{2^3}$$ (1 mark)

$\therefore$ S is geometric with common ratio $r = \frac{1}{2^3}$ (1 mark)

[3 marks]

(ii) $$S_n = \frac{\frac{1}{2}\left[1 - \left(\frac{1}{2}\right)^{3n}\right]}{1 - \left(\frac{1}{2}\right)^3}$$ (1 mark)

$$= \frac{\frac{1}{2}\left[1 - \frac{1}{2^{3n}}\right]}{1 - \frac{1}{8}}$$ (1 mark)

$$= \frac{1}{2} \times \frac{8}{7}\left[1 - \frac{1}{2^{3n}}\right]$$ (1 mark)

$$= \frac{4}{7}\left[1 - \frac{1}{2^{3n}}\right]$$ (1 mark)

[4 marks]

(b) (i) $f(x) = \cos 2x \quad \Rightarrow \quad f^{1}(x) = -2\sin\ 2x$ (1 mark)

$\Rightarrow \quad f^{11}(x) = -4\cos 2x$ (1 mark)

$\Rightarrow \quad f^{111}(x) = 8\ \sin 2x$ (1 mark)

$\Rightarrow \quad f^{iv}(x) = 16\ \cos 2x$ (1 mark)

so, $f(0) = 1,\ \ f^{1}(0) = 0,\ \ f^{11}(x) = -4,\ \ f^{111}(0) = 0,\ \ f^{iv}(0) = 16$ (1 mark)

Hence, by Maclaurin's Theorem,

$\cos 2x\ =\ 1\ - \dfrac{4x^2}{2!} + \dfrac{16x^4}{4!} -$ (1 mark)

$= 1 - 2x^2 + \dfrac{2}{3}x^4$ (1 mark)

[7 marks]

(c) (i) $\sqrt{\left(\dfrac{1+x}{1-x}\right)}$

$=\ (1+x)^{½}\ (1-x)^{-½}$ (1 mark)

$=\ \left(1 + \dfrac{1}{2}x - \dfrac{1}{8}x^2 + \dfrac{1}{16}x^3 \ldots\right)\left(1 + \dfrac{1}{2}x + \dfrac{3}{8}x^2 + \dfrac{5}{16}x^3 \ldots\right)$ (3 marks)

$=\ 1 + x + \dfrac{1}{2}x^2 + \dfrac{1}{2}x^3$ (2 marks)

for $-1 < x < 1$ (1 mark)

[7 marks]

(ii) $\sqrt{\dfrac{1.02}{0.98}} = \sqrt{\dfrac{102}{98}} = \dfrac{1}{7}\sqrt{51}$ (1 mark)

$\sqrt{51}\ =\ 7\sqrt{\dfrac{1+x}{1-x}}$ where $x = 0.02$ (1 mark)

$\Rightarrow\ \sqrt{51}\ =\ 7\ \left\{1 + 0.02 + \dfrac{1}{2}(0.02)^2 + \dfrac{1}{2}(0.02)^3\right\}$ (1 mark)

$= 7.14141$ (5 d.p.) (1 mark)

[4 marks]

Specific Objectives: (B) 5, 9, 11; (C)3, 4

Total 25 marks

SECTION C

(MODULE 3)

Question 5

(a) (i) P (First card drawn has even number) $= \frac{5}{10} = \frac{1}{2}$ (1 mark)

P (Second card drawn has even number) $= \frac{4}{9}$ (2 marks)

$\therefore$ P (Both cards have even numbers) $= \left(\frac{1}{2}\right)\left(\frac{4}{9}\right)$

$= \frac{2}{9}$ (1 mark)

[4 marks]

(ii) P (Both cards have odd numbers) $= \frac{2}{9}$ (1 mark)

P (One card has odd and the other has even i.e. both cards do not have odd or do not have even numbers) $= 1 - 2\left(\frac{2}{9}\right)$ (2 marks)

$= \frac{5}{9}$ (1 mark)

[3 marks]

(b) (i) a) $\frac{82}{150} = 0.547$ [2 marks]

ii) $\frac{39}{150} + \frac{75}{150} = 0.76$ [4 marks]

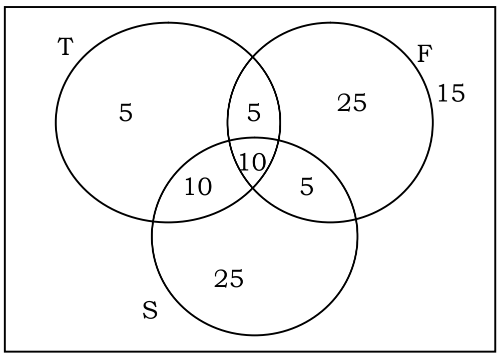
iii) $\frac{82}{150} + \frac{39}{150} - \frac{27}{150} = 0.267$ [3 marks]

(c) Let T, S and F represent respectively the customers purchasing tools, seeds and fertilizer.

(i) One mark for any two correct numbers (4 marks)

(ii) a) 5 (1 mark)
b) 10 (1 mark)
c) 5 (1 mark)
d) 15 (1 mark)

(i)



Venn diagram

Total 25 marks

Specific Objectives: (A) 5, 6, 7, 9, 10, 11, 12, 13

Question 6

(a) $\begin{vmatrix} 5 & x & 3 \\ x+2 & 2 & 1 \\ -3 & 2 & x \end{vmatrix} = 0$

$5\,(2x - 2) - x\,(x^2 + 2x + 3) + 3\,(2x + 4 + 6) = 0$ (3 marks)

$x^3 + 2x^2 - 13x - 20 = 0$ (1 mark)

Subs $x = -4$, $(-4)^3 + 2\,(-4)^2 - 13(-4) - 20 = 0$

$(x+4)\,(x^2 - 2x - 5) = 0$ (2 marks)

$x = -4$

$x = \dfrac{2 \pm \sqrt{24}}{2}$

$x = 1 \pm \sqrt{6}$ (2 marks)

[8 marks]

Alternatively

$\begin{vmatrix} 5 & x & 3 \\ x+2 & 2 & 1 \\ -3 & 2 & x \end{vmatrix} = 0$ □ $\begin{vmatrix} 4+x & x+4 & x+4 \\ x+2 & 2 & 1 \\ -3 & 2 & x \end{vmatrix} = 0$ (Add rows 2 and 3 to row 1)

□ $(x+4)\begin{vmatrix} 1 & 1 & 1 \\ x+2 & 2 & 1 \\ -3 & 2 & x \end{vmatrix} = 0$ □ $(x+4)\begin{vmatrix} 0 & 1 & 0 \\ x & 2 & -1 \\ -5 & 2 & x-2 \end{vmatrix} = 0$

(subtract columns 2 from Columns 1 and 3).

□ $(x+4)x - (x^2 - 2x - 5) = 0$ □ $x = -4$ or $1 \pm \sqrt{6}$ [8 marks]

(b) $y \tan x \dfrac{\mathrm{d}y}{\mathrm{d}x} = (4 + y^2)\sec^2 x$ (1 mark)

$\dfrac{y}{4+y^2}\,\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\sec^2 x}{\tan x}$

$\dfrac{y\,\mathrm{d}y}{4+y^2} = \dfrac{\sec^2 x\,\mathrm{d}x}{\tan x}$

$\displaystyle\int \frac{y\,\mathrm{d}y}{4+y^2} = \int \frac{\sec^2 x\,\mathrm{d}x}{\tan x}$ (1 mark)

$\frac{1}{2}\ln(4 + y^2) = \ln|\tan x| + c$ (2 marks)

[4 marks]

(c) (i) $y = u\cos 3x + v\sin 3x$

□ $\frac{dy}{dx} = -3u\sin 3x + 3v\cos 3x$ (2 marks)

□ $\frac{d^2y}{dx^2} = -9u\cos 3x - 9v\sin 3x$ (2 marks)

so, $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = -30\sin 3x$

□ $-(6v - 12u)\sin 3x + (-6u + 12v)\cos 3x = -30\sin 3x$ (1 mark)

□ $2u + v = 5$ and $u = 2v$ (1 mark)

□ $u = 2$ and $v = 1$ (2 marks)

[8 marks]

(ii) the auxiliary equation of the different equation is

$k^2 + 4k + 3 = 0$ (1 mark)

□ $(k + 3)(k + 1) = 0$

□ $k = -3$ or -1 (2 marks)

□ the complementary function is

$y = Ae^{-x} + Be^{-3x}$; where A, B are constants (1 mark)

General solution is $y = Ae^{-x} + Be^{-3x} + \sin 3x + 2\cos 3x$ (1 mark)

[5 marks]

Total 25 marks

Specific Objectives: (B) 1, 5, 6; (C) 1, 3

End of Test